LEVEL M

M 1-30 : Points & Lines

Distance Formula

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Also, the distance between origin O and point $A(x_1, y_1)$

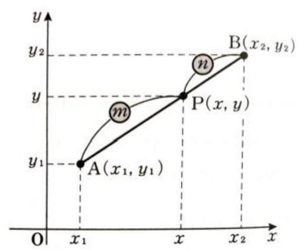
$$OA=\sqrt{{x_1}^2+{y_1}^2}$$

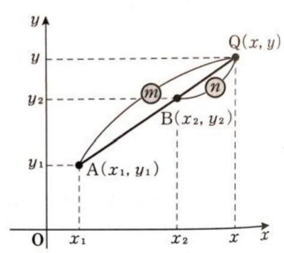
Coordinates of Internal/External Dividing Points

Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the coordinates of the points that divide line segment AB in the ratio $m : n$ are as follows:

Internally, $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}\right)$

Externally, $\left(\frac{-nx_1+mx_2}{m-n}, \frac{-ny_1+my_2}{m-n}\right)$





Note: The midpoint of line segment AB can be found by taking m=1 and n=1.

Centre of Gravity of Triangles

Given $\triangle ABC$ with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, the coordinates of the centre of gravity G are given by

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

The **equation of a line** can be expressed in:

Form 1: Point-gradient form (m is gradient)

$$y-y_1=m(x-x_1)$$

Form 2: Gradient-Intercept form

$$y=mx+b$$

Form 3: Standard form

$$ax+by+c=0$$

Form 4: Intercept form

$$\frac{x}{a}+\frac{y}{b}=1$$

a and b are the x-intercept and y-intercept, $ab \neq 0$

Two lines $y=m_1x+n$ and $y=m_2x+n$ are

- **parallel** if $\boldsymbol{m_1=m_2}$
- **perpendicular** if $\boldsymbol{m_1 \cdot m_2=-1}$

The point at which all three perpendiculars of a triangle dropped from the vertices intersect is called the ***orthocentre*** of a triangle.

The point at which all the perpendicular bisectors of three sides of a triangle intersect is called the ***circumcentre*** of a triangle.

Distance from a Point to a Line

The distance d from point (x_1, y_1) to line $ax+by+c=0$ is

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Refer to the worksheets for more examples.

M 31-50 : Circles

The **equation of a circle** can be expressed in:

Form 1: Centre (a, b) and radius r

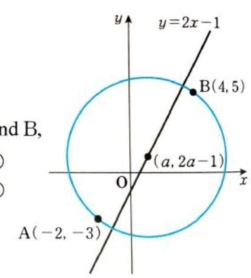
$$(x-a)^2+(y-b)^2=r^2$$

Form 2: Centre and radius unknown

$$x^2+y^2+ax+by+c=0$$

Find the equation of the circle passing through two points $A(-2, -3)$ and $B(4, 5)$ with its centre on line $y=2x-1$.

Let the centre of the circle be $(a, 2a-1)$.
Let the radius of the circle be r.
$(x-a)^2+[y-(2a-1)]^2=r^2$
Since this circle passes through points A and B,
$(-2-a)^2+[-3-(2a-1)]^2=r^2$ …①
$(4-a)^2+[5-(2a-1)]^2=r^2$ …②
From ① and ②, $a=1$, $r^2=25$
$\therefore\ (x-1)^2+(y-1)^2=25$



When a quadratic equation $ax^2+bx+c=0$ is obtained after y is eliminated from solving simultaneous equations of a line and a circle, let the discriminant be $D=b^2-4ac$.

$D>0 \Leftrightarrow$ They intersect at two distinct points

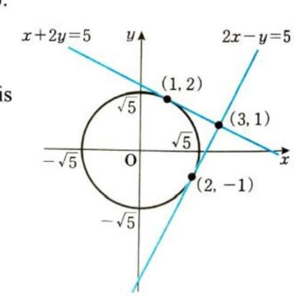
$D=0 \Leftrightarrow$ They are tangent and touch at one point

$D<0 \Leftrightarrow$ They do not intersect

The **equation of the tangent** to circle $x^2+y^2=r^2$ at (x_1, y_1) is $\boldsymbol{x_1x+y_1y=r^2}$

Find the equation of the line which passes through point $(3, 1)$ and is tangent to circle $x^2+y^2=5$.

Let the tangent point be (a, b).
$a^2+b^2=5$ …①
Also, the equation of the tangent is
$ax+by=5$ …②
Since the tangent passes through point $(3, 1)$, $3a+b=5$ …③
From ① and ③, $a^2-3a+2=0$
$\therefore\ a=1, 2$; $b=2, -1$
Therefore, $x+2y=5$, $2x-y=5$



Harmonic Form in Trigonometry

$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha)$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$\begin{aligned}\sqrt{3}\sin\theta+\cos\theta &= 2\left(\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\right)\\ &= 2\left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)\\ &= \mathbf{2\sin\left(\theta+\frac{\pi}{6}\right)}\end{aligned}$$

Applications (equations, inequalities, max/min):

Given $0 \leqslant \theta < 2\pi$, solve the equation $\sin\theta+\sqrt{3}\cos\theta = -\sqrt{2}$

$$\begin{aligned}\sin\theta+\sqrt{3}\cos\theta &= 2\left(\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta\right)\\ &= 2\left(\sin\theta\cos\frac{\pi}{3}+\cos\theta\sin\frac{\pi}{3}\right) = 2\sin\left(\theta+\frac{\pi}{3}\right)\end{aligned}$$

$\therefore\ 2\sin\left(\theta+\frac{\pi}{3}\right) = -\sqrt{2}$

So, $\sin\left(\theta+\frac{\pi}{3}\right) = -\frac{\sqrt{2}}{2}$

Since $0 \leqslant \theta < 2\pi$, $\frac{\pi}{3} \leqslant \theta+\frac{\pi}{3} < \frac{7}{3}\pi$

$\theta+\frac{\pi}{3} = \frac{5}{4}\pi,\ \frac{7}{4}\pi \quad \therefore\ \boldsymbol{\theta = \frac{11}{12}\pi,\ \frac{17}{12}\pi}$

$\sqrt{3}\sin\theta-\cos\theta \leqslant \sqrt{2}$ $\quad \boxed{0 \leqslant \theta < 2\pi}$

$$\begin{aligned}\sqrt{3}\sin\theta-\cos\theta &= 2\left(\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta\right)\\ &= 2\left[\sin\theta\cos\left(-\frac{\pi}{6}\right)+\cos\theta\sin\left(-\frac{\pi}{6}\right)\right] = 2\sin\left(\theta-\frac{\pi}{6}\right)\end{aligned}$$

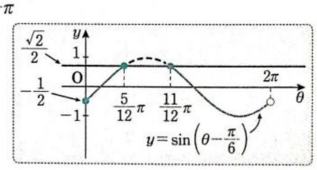
Therefore, $2\sin\left(\theta-\frac{\pi}{6}\right) \leqslant \sqrt{2}$, i.e. $\sin\left(\theta-\frac{\pi}{6}\right) \leqslant \frac{\sqrt{2}}{2}$

Since $0 \leqslant \theta < 2\pi$, $-\frac{\pi}{6} \leqslant \theta-\frac{\pi}{6} < \frac{11}{6}\pi$

When $\sin\left(\theta-\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$,

$\therefore\ \theta = \frac{5}{12}\pi,\ \frac{11}{12}\pi$

$\therefore\ \boldsymbol{0 \leqslant \theta \leqslant \frac{5}{12}\pi,\ \frac{11}{12}\pi \leqslant \theta < 2\pi}$



Product-to-Sum Formulas

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

Sum-to-Product Formulas

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

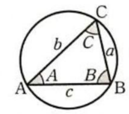
Note: Refer to the worksheets for applications.

M 181-190 : Sine & Cosine Rules

Sine Rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

(R is the radius of the circumscribed circle of $\triangle$ABC)



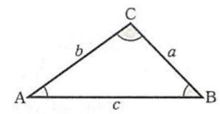
Cosine Rule

Given $\triangle$ABC,

$a^2 = b^2+c^2-2bc\cos A$

$b^2 = c^2+a^2-2ca\cos B$

$c^2 = a^2+b^2-2ab\cos C$



Examples:

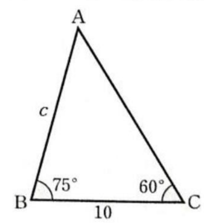
Given $\triangle$ABC where $a=10$, $B=75°$ and $C=60°$, find c.

$A = 180° - (75°+60°) = 45°$

$\frac{10}{\sin 45°} = \frac{c}{\sin 60°}$

$\therefore\ c = \frac{10\sin 60°}{\sin 45°}$

$= 5\sqrt{6}$



Given $\triangle$ABC where $a=\sqrt{7}$, $b=\sqrt{3}$ and $A=150°$, find c.

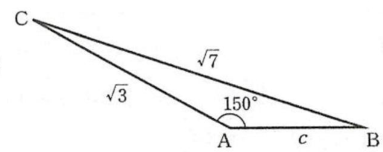
$(\sqrt{7})^2 = (\sqrt{3})^2+c^2-2\cdot\sqrt{3}\cdot c\cos 150°$

$\therefore\ c^2+3c-4=0$

$(c+4)(c-1)=0$

$\therefore\ c=-4,\ 1$

Since $c>0$, $\boldsymbol{c=1}$



M 191-200 : Area of Triangles

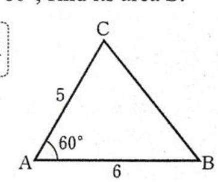
Area of a Triangle

$S = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

Given $\triangle$ABC where $b=5$, $c=6$ and $A=60°$, find its area S.

$S = \frac{1}{2}\cdot 5\cdot 6\sin 60°$ $\quad \boxed{S=\frac{1}{2}bc\sin A}$

$= \mathbf{\frac{15\sqrt{3}}{2}}$



Additional Formulas:

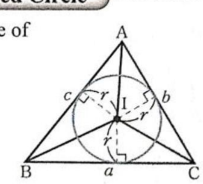
Area of a Triangle with an Inscribed Circle

Let S be the area of $\triangle$ABC, I be the centre of the inscribed circle, and r be the radius.

$S = \triangle\text{IBC}+\triangle\text{ICA}+\triangle\text{IAB}$

$= \frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr$

$= \frac{1}{2}r(a+b+c)$



It is also possible to find the area of a triangle using ***Heron's Formula*** (revisited in level XT).

M 151-180 : Addition Formulas

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

$$\sin 165^\circ = \sin(120^\circ + 45^\circ) = \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1\cdot\frac{\sqrt{3}}{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}} = \mathbf{2-\sqrt{3}}$$

More examples:

Let $\frac{\pi}{2} < \alpha < \pi$ and $\pi < \beta < \frac{3}{2}\pi$. Find the value of $\sin(\alpha-\beta)$ when $\sin\alpha = \frac{3}{5}$ and $\cos\beta = -\frac{12}{13}$.

Since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$ $\quad \cos\alpha = -\sqrt{1-\left(\frac{3}{5}\right)^2} = -\frac{4}{5}$

Since $\pi < \beta < \frac{3}{2}\pi$, $\sin\beta < 0$ $\quad \sin\beta = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$

$$\sin(\alpha-\beta) = \frac{3}{5}\cdot\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{13}\right) = -\frac{\mathbf{56}}{\mathbf{65}}$$

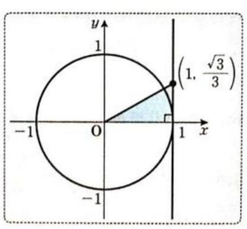
Let $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$. Find the value of $\alpha+\beta$ when $\tan\alpha = \frac{\sqrt{3}}{7}$ and $\tan\beta = \frac{\sqrt{3}}{6}$.

$$\tan(\alpha+\beta) = \frac{\frac{\sqrt{3}}{7} + \frac{\sqrt{3}}{6}}{1 - \frac{\sqrt{3}}{7}\cdot\frac{\sqrt{3}}{6}} = \frac{\sqrt{3}}{3}$$

Since $0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$,

$0 < \alpha+\beta < \pi$

$\therefore\ \boldsymbol{\alpha+\beta = \frac{\pi}{6}}$



Double-angle Formulas

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

Triple-angle Formulas

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

Half-angle Formulas

$\sin^2\frac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$ $\qquad \tan^2\frac{\alpha}{2} = \dfrac{1-\cos\alpha}{1+\cos\alpha}$

$\cos^2\frac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$

Let $\frac{\pi}{2} < \alpha < \pi$. Find the values of $\cos 2\alpha$ and $\sin 2\alpha$ when $\sin\alpha = \frac{2}{5}$.

$\cos 2\alpha = 1 - 2\cdot\left(\frac{2}{5}\right)^2 = \frac{\mathbf{17}}{\mathbf{25}}$

Also, since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$

$\cos\alpha = -\sqrt{1-\left(\frac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}$

$\therefore\ \sin 2\alpha = 2\cdot\frac{2}{5}\cdot\left(-\frac{\sqrt{21}}{5}\right) = -\frac{\mathbf{4\sqrt{21}}}{\mathbf{25}}$

Let $\pi < \alpha < \frac{3}{2}\pi$. Find the value of $\sin\frac{\alpha}{2}$ when $\sin\alpha = -\frac{2\sqrt{2}}{3}$.

Since $\pi < \alpha < \frac{3}{2}\pi$, $\cos\alpha < 0$

$\cos\alpha = -\sqrt{1-\left(-\frac{2\sqrt{2}}{3}\right)^2} = -\frac{1}{3}$

$\therefore\ \sin^2\frac{\alpha}{2} = \dfrac{1-\left(-\frac{1}{3}\right)}{2} = \frac{2}{3}$

Since $\pi < \alpha < \frac{3}{2}\pi$, $\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3}{4}\pi$; therefore, $\sin\frac{\alpha}{2} > 0$

$\therefore\ \boldsymbol{\sin\frac{\alpha}{2} = \frac{\sqrt{6}}{3}}$

We can apply the formulas to solving equations/ inequalities, and finding maxima/minima.

In these examples, the domain is $\boldsymbol{0 \le \theta < 2\pi}$.

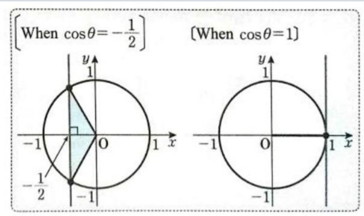
$\cos 2\theta - \cos\theta = 0$

$(2\cos^2\theta - 1) - \cos\theta = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\cos\theta = -\frac{1}{2},\ 1$

$\therefore\ \boldsymbol{\theta = 0,\ \frac{2}{3}\pi,\ \frac{4}{3}\pi}$



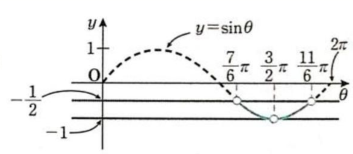
$\cos 2\theta - 3\sin\theta - 2 > 0$

$(1 - 2\sin^2\theta) - 3\sin\theta - 2 > 0$

$2\sin^2\theta + 3\sin\theta + 1 < 0$

$-1 < \sin\theta < -\frac{1}{2}$

$\therefore\ \boldsymbol{\frac{7}{6}\pi < \theta < \frac{3}{2}\pi,\ \frac{3}{2}\pi < \theta < \frac{11}{6}\pi}$



Given $0 \leqslant \theta < 2\pi$, find the maximum and minimum values of function $y = 2\sin\theta + \cos 2\theta + 1$ and state the corresponding values of θ.

$y = 2\sin\theta + (1 - 2\sin^2\theta) + 1 = -2\sin^2\theta + 2\sin\theta + 2$

Let $\sin\theta = t$. $-1 \leqslant t \leqslant 1$

$y = -2t^2 + 2t + 2$

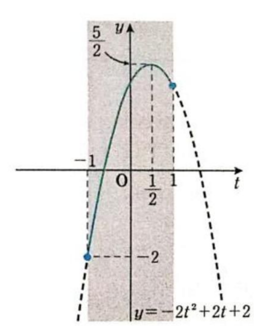
$\quad = -2\left(t - \frac{1}{2}\right)^2 + \frac{5}{2}$

the maximum value is $\boldsymbol{\frac{5}{2}}$,

at $t = \sin\theta = \frac{1}{2}$, i.e. $\boldsymbol{\theta = \frac{\pi}{6},\ \frac{5}{6}\pi}$

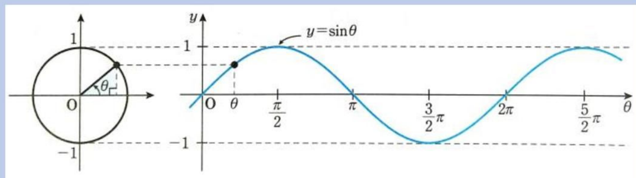
the minimum value is $\boldsymbol{-2}$,

at $t = \sin\theta = -1$, i.e. $\boldsymbol{\theta = \frac{3}{2}\pi}$.

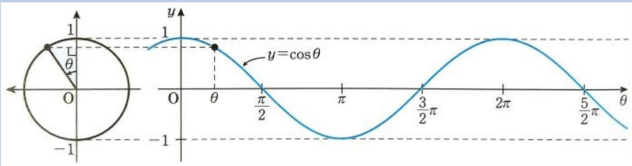


M 131-140 : Graphs of Trigonometric Functions

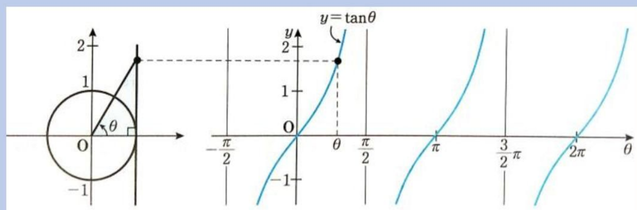
Graph of $\boldsymbol{y=\sin\theta}$, range: $-1 \le \sin\theta \le 1$



Graph of $\boldsymbol{y=\cos\theta}$, range: $-1 \le \cos\theta \le 1$



Graph of $\boldsymbol{y=\tan\theta}$ $\left(\theta \neq \frac{\pi}{2}+n\pi\right)$, range: all $\mathbb{R}$



Period is the minimum interval/distance until the function repeats itself.

Function, $\boldsymbol{y=}$	$\sin k\theta$	$\cos k\theta$	$\tan k\theta$
Period	$\dfrac{2\pi}{k}$	$\dfrac{2\pi}{k}$	$\dfrac{\pi}{k}$

The graph of $\boldsymbol{y=A\sin k(\theta-p)+q}$ is a translation of the graph of $\boldsymbol{y=A\sin k\theta}$, p units along the θ-axis and q units along the y-axis.

Similarly for cosine and tangent functions.

M 141-150 : Trigonometric Inequalities

We solve a trigonometric inequality **graphically**.

1) Rearrrange the inequality in the form $\sin\theta < k$

2) Draw the graph of the trigonometric function (take note of the domain) and solve the *equation*.

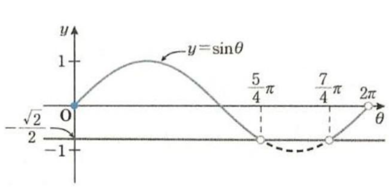
3) Draw a horizontal line $y=k$. Highlight the region satisfying the *inequality*.

$2\sin\theta+\sqrt{2}>0$

$\sin\theta > -\frac{\sqrt{2}}{2}$

When $\sin\theta=-\frac{\sqrt{2}}{2}$,

$\theta=\frac{5}{4}\pi,\ \frac{7}{4}\pi$



$\boldsymbol{0 \leqslant \theta < \frac{5}{4}\pi,\ \frac{7}{4}\pi < \theta < 2\pi}$

$-2<\sqrt{3}\tan\theta+1<2$

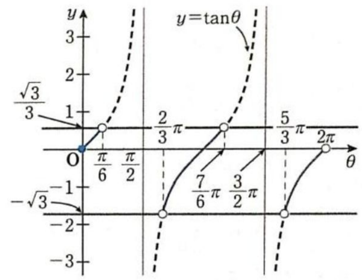
$-\sqrt{3}<\tan\theta<\frac{\sqrt{3}}{3}$

When $\tan\theta=-\sqrt{3}$,

$\theta=\frac{2}{3}\pi,\ \frac{5}{3}\pi$

When $\tan\theta=\frac{\sqrt{3}}{3}$,

$\theta=\frac{\pi}{6},\ \frac{7}{6}\pi$



$\boldsymbol{0 \leqslant \theta < \frac{\pi}{6},\ \frac{2}{3}\pi < \theta < \frac{7}{6}\pi,\ \frac{5}{3}\pi < \theta < 2\pi}$

In the above examples, the domain is $\boldsymbol{0 \le \theta < 2\pi}$.

Sometimes, we may need to transform the domain.

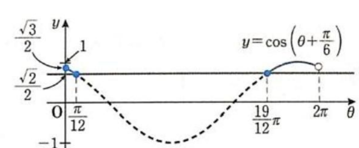
$\cos\left(\theta+\frac{\pi}{6}\right) \geqslant \frac{\sqrt{2}}{2}$

Since $0 \leqslant \theta < 2\pi$, $\frac{\pi}{6} \leqslant \theta+\frac{\pi}{6} < \frac{13}{6}\pi$

When $\cos\left(\theta+\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$,

$\theta+\frac{\pi}{6}=\frac{\pi}{4},\ \frac{7}{4}\pi$

$\therefore\ \theta=\frac{\pi}{12},\ \frac{19}{12}\pi$



$\boldsymbol{0 \leqslant \theta \leqslant \frac{\pi}{12},\ \frac{19}{12}\pi \leqslant \theta < 2\pi}$

Variations: Use trigonometric identities to form a quadratic inequality in $\sin\theta$ / $\cos\theta$ / $\tan\theta$.

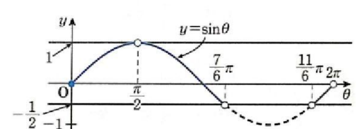
$2\cos^2\theta+\sin\theta-1>0$

$2(1-\sin^2\theta)+\sin\theta-1>0$

$2\sin^2\theta-\sin\theta-1<0$

$(2\sin\theta+1)(\sin\theta-1)<0$

$\therefore\ -\frac{1}{2}<\sin\theta<1$



When $\sin\theta=-\frac{1}{2}$, $\theta=\frac{7}{6}\pi,\ \frac{11}{6}\pi$ When $\sin\theta=1$, $\theta=\frac{\pi}{2}$

$\boldsymbol{0 \leqslant \theta < \frac{\pi}{2},\ \frac{\pi}{2} < \theta < \frac{7}{6}\pi,\ \frac{11}{6}\pi < \theta < 2\pi}$

Maxima and Minima of Trigonometric Functions

Let the basic trigonometric function be t. First determine the range of values of t, and then find the range of the quadratic function in t.

Given $0 \leqslant \theta < 2\pi$, find the maximum and minimum values of function $y=-\cos^2\theta-\sin\theta+3$ and state the corresponding values of θ.

$y=-(1-\sin^2\theta)-\sin\theta+3$

$=\sin^2\theta-\sin\theta+2$

Let $\sin\theta=t$. $-1 \leqslant t \leqslant 1$

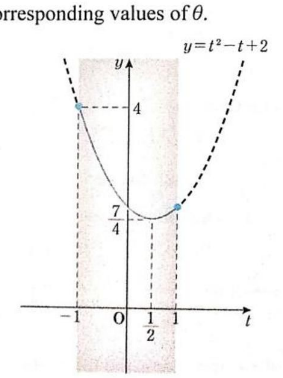
$y=t^2-t+2=\left(t-\frac{1}{2}\right)^2+\frac{7}{4}$

the maximum value is 4,

at $t=\sin\theta=-1$, **i.e.** $\boldsymbol{\theta=\frac{3}{2}\pi}$

the minimum value is $\frac{7}{4}$,

at $t=\sin\theta=\frac{1}{2}$, **i.e.** $\boldsymbol{\theta=\frac{\pi}{6},\ \frac{5}{6}\pi}$.



$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$

M 101-120 : Properties of Trigonometric Functions

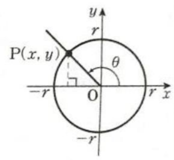
For any general angle θ drawn **counter-clockwise** from the **positive *x*-axis**, we have the following.

Definitions of Trigonometric Functions

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$

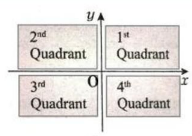
Also, when the length of radius r of the circle is 1,

$\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$

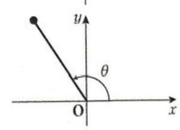


We define four quadrants on the Cartesian plane.

The four regions on a coordinate plane that are separated by the x-axis and y-axis are called the 1st ***Quadrant***, 2nd ***Quadrant***, 3rd ***Quadrant*** and 4th ***Quadrant*** respectively as shown in the diagram on the right.



When the terminal side of θ is in the 2nd Quadrant, θ is called an ***angle in the 2nd Quadrant***.



Trigonometric Identities:

$$\left.\begin{aligned} \sin(\theta + 360° \times n) &= \sin\theta \\ \cos(\theta + 360° \times n) &= \cos\theta \\ \tan(\theta + 180° \times n) &= \tan\theta \end{aligned}\right\} \textit{Periodic Identities}$$

$$\left.\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \tan(-\theta) &= -\tan\theta \end{aligned}\right\} \textit{Odd/ Even Identities}$$

~~~~~~

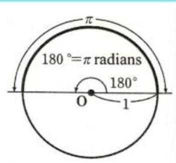
We can also measure an angle in **radians**.

Circular Measure

Since $180° = \pi$ radians,

$1° = \frac{\pi}{180}$ radians, $\quad$ 1 radian $= \frac{180°}{\pi}$

Generally, the unit "radians" is omitted.



E.g. $30° = \frac{\pi}{6}$, $75° = \frac{5\pi}{12}$, $90° = \frac{\pi}{2}$, $120° = \frac{2\pi}{3}$

Note: You *do not* need to memorise every trigonometric identity. There are generally two ways to evaluate unfamiliar trigonometric functions.

Method 1: Use the known identities above

$$\begin{aligned} \sin(\theta + 90°) &= \sin(90° - (-\theta)) \\ &= \cos(-\theta) \quad \textit{cofunction identity} \\ &= \cos(\theta) \quad \textit{even identity} \end{aligned}$$

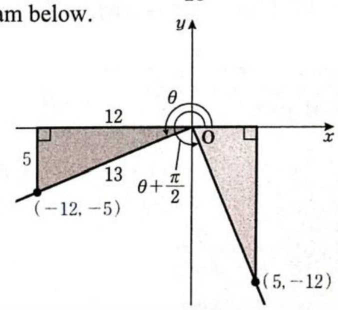
Method 2: Determine the functions with a diagram

θ is an angle in the 3rd Quadrant and $\cos\theta = -\frac{12}{13}$, evaluate the expressions using the diagram below.

$\sin\left(\theta + \frac{\pi}{2}\right) = -\frac{12}{13}$

$\cos\left(\theta + \frac{\pi}{2}\right) = \frac{5}{13}$

$\tan\left(\theta + \frac{\pi}{2}\right) = -\frac{12}{5}$



M 121-130 : Trigonometric Equations

Use the known trigonometric functions/ratios.

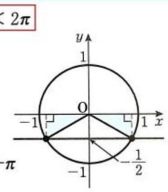
Given $0 \leqslant \theta < 2\pi$

$2\sin\theta + 1 = 0$

$\sin\theta = -\frac{1}{2}$

Therefore,

$\theta = \frac{7}{6}\pi, \ \frac{11}{6}\pi$

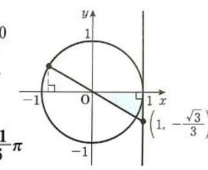


$3\tan\theta + \sqrt{3} = 0$

$\tan\theta = -\frac{\sqrt{3}}{3}$

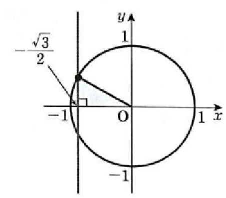
Therefore,

$\theta = \frac{5}{6}\pi, \ \frac{11}{6}\pi$



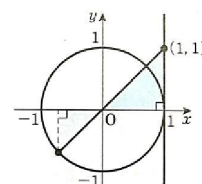
$\cos\theta = -\frac{\sqrt{3}}{2} \quad (0 \leqslant \theta < \pi)$

$\theta = \frac{5}{6}\pi$



$\tan\theta = 1 \quad (0 \leqslant \theta < 4\pi)$

$\theta = \frac{\pi}{4}, \ \frac{5}{4}\pi, \ \frac{9}{4}\pi, \ \frac{13}{4}\pi$



In some case, we may need to transform the domain.

$\sqrt{2}\sin\left(2\theta + \frac{\pi}{3}\right) - 1 = 0$

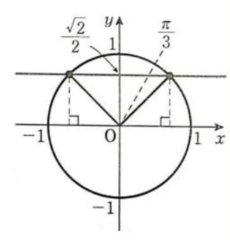
$\sin\left(2\theta + \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

Given $0 \leqslant \theta < 2\pi$, $\quad \overbrace{\frac{\pi}{3} \leqslant 2\theta + \frac{\pi}{3} < \frac{13}{3}\pi}^{\text{Transforming the domain}}$

Therefore,

$2\theta + \frac{\pi}{3} = \frac{3}{4}\pi, \ \frac{9}{4}\pi, \ \frac{11}{4}\pi, \ \frac{17}{4}\pi$

$\therefore \ \theta = \frac{5}{24}\pi, \ \frac{23}{24}\pi, \ \frac{29}{24}\pi, \ \frac{47}{24}\pi$



More Variations: Use trigonometric identities to obtain a quadratic equation in $\sin\theta$ / $\cos\theta$. Solve the quadratic and then the trigonometric equation(s).

$2\sin^2\theta + \cos\theta - 1 = 0$

$2(1 - \cos^2\theta) + \cos\theta - 1 = 0$

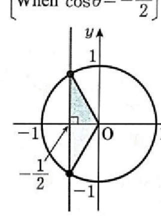
$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

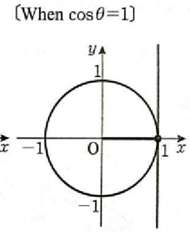
$\cos\theta = -\frac{1}{2}, \ 1$

$\therefore \ \theta = 0, \ \frac{2}{3}\pi, \ \frac{4}{3}\pi$

[When $\cos\theta = -\frac{1}{2}$]



[When $\cos\theta = 1$]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M 71-80 : Regions (and Inequalities)

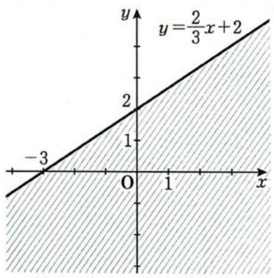
1) Draw the line/curve representing the *equation*.

2) Shade the region representing the *inequality* (upper/lower region ; inner/outer region)

3) Specify if the boundary should be included.

$2x-3y>-6$

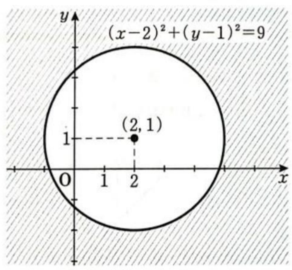
Rearranging, $y<\frac{2}{3}x+2$



The boundary is not included.

$x^2-4x-4>-y^2+2y$

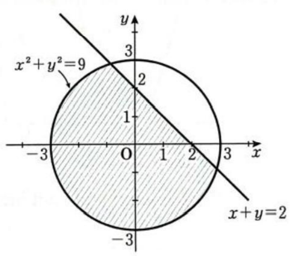
Rearranging, $(x-2)^2+(y-1)^2>9$



The boundary is not included.

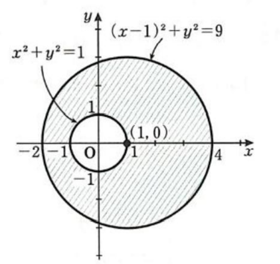
When there are multiple inequalities/constraints,

$$\begin{cases} x^2+y^2\leqslant 9 \\ x+y\leqslant 2 \end{cases}$$



The boundaries are included.

$$\begin{cases} x^2+y^2>1 \\ (x-1)^2+y^2<9 \end{cases}$$



The boundaries are not included.

For real numbers a and b, $\boldsymbol{ab>0}$ implies that either $\boldsymbol{a>0, b>0}$ or $\boldsymbol{a<0, b<0}$. Similarly, $\boldsymbol{ab<0}$ implies that either $\boldsymbol{a>0, b<0}$ or $\boldsymbol{a<0, b>0}$.

$(y+x^2)(y-x^2+2)>0$

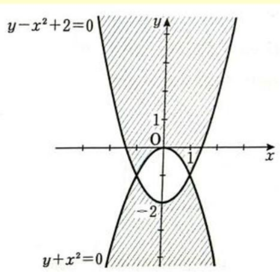
When the given inequality is true, either

$$\begin{cases} y+x^2>0 \\ y-x^2+2>0 \end{cases}$$

or

$$\begin{cases} y+x^2<0 \\ y-x^2+2<0 \end{cases}$$

is true.



The boundaries are not included.

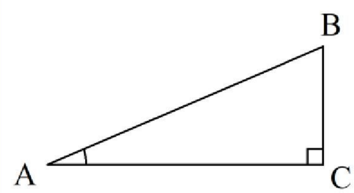
<u>More variations:</u> Linear and nonlinear programming

Problem: Maximise/minimise an objective function

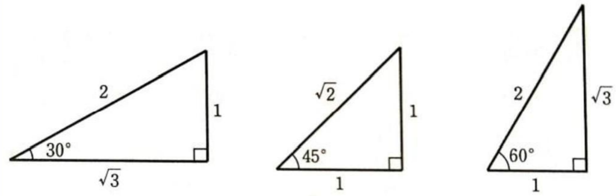
Constraint: The graph of the objective function can only move within the feasible region

Refer to M 77-79 for elaborated examples.

M 81-100 : Trigonometric Ratios



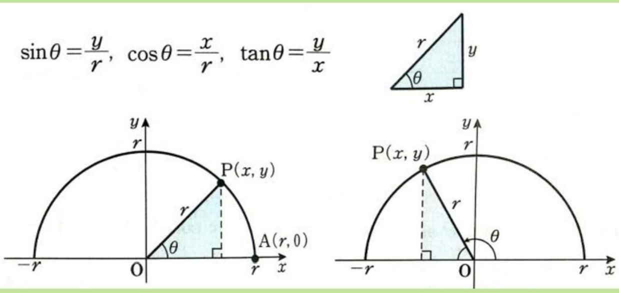
$$\sin A=\frac{BC}{AB}\qquad \cos A=\frac{AC}{AB}\qquad \tan A=\frac{BC}{AC}$$



$$\begin{cases} \sin 30^\circ=\frac{1}{2} \\ \cos 30^\circ=\frac{\sqrt{3}}{2} \\ \tan 30^\circ=\frac{\sqrt{3}}{3} \end{cases}\quad \begin{cases} \sin 45^\circ=\frac{\sqrt{2}}{2} \\ \cos 45^\circ=\frac{\sqrt{2}}{2} \\ \tan 45^\circ=1 \end{cases}\quad \begin{cases} \sin 60^\circ=\frac{\sqrt{3}}{2} \\ \cos 60^\circ=\frac{1}{2} \\ \tan 60^\circ=\sqrt{3} \end{cases}$$

Draw a semicircle with the centre at origin O and radius r, and let A be point $(r, 0)$. Place point P (x, y) on the circumference of this semicircle and let $\angle$ AOP $=\theta$ and $0^\circ\leq\theta\leq 180^\circ$. Then

$$\sin\theta=\frac{y}{r},\ \cos\theta=\frac{x}{r},\ \tan\theta=\frac{y}{x}$$



Note that we have a unit circle when $r = 1$.

Moreover, we have

$$\begin{cases} \sin 0^\circ=0 \\ \cos 0^\circ=1 \\ \tan 0^\circ=0 \end{cases}\quad \begin{cases} \sin 90^\circ=1 \\ \cos 90^\circ=0 \\ \tan 90^\circ=\infty \end{cases}\quad \begin{cases} \sin 180^\circ=0 \\ \cos 180^\circ=-1 \\ \tan 180^\circ=0 \end{cases}$$

<u>**Trigonometric Identities:**</u>

$$\tan\theta=\frac{\sin\theta}{\cos\theta}\ \text{①}\quad,\quad \sin^2\theta+\cos^2\theta=1\ \text{②}$$

$$\tan^2\theta+1=\frac{1}{\cos^2\theta}\ \text{③}$$

(1) *Quotient Identity* ; (2),(3) *Pythagorean Identity*

$$\left.\begin{aligned} \sin(90^\circ-\theta)&=\cos\theta \\ \cos(90^\circ-\theta)&=\sin\theta \\ \tan(90^\circ-\theta)&=\frac{1}{\tan\theta} \end{aligned}\right\}\ \textit{Cofunction Identities}$$

The radius of a circle is equal to the perpendicular distance from the centre of the circle to any tangent.

Find the equation of the line which passes through point $(3, 4)$ and is tangent to circle $(x+1)^2+(y-2)^2=10$.

Let the gradient of the tangent be m. The equation is $y-4=m(x-3)$

So, $mx-y-3m+4=0$ …①

Let the distance from the centre $(-1, 2)$ to line ① be d.

$$d=\frac{|m\cdot(-1)-2-3m+4|}{\sqrt{m^2+(-1)^2}}=\frac{|4m-2|}{\sqrt{m^2+1}}$$

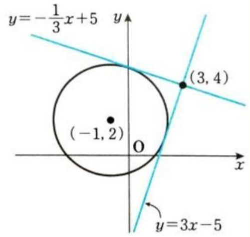
Since distance d equals to the radius,

$$\frac{|4m-2|}{\sqrt{m^2+1}}=\sqrt{10} \quad \cdots ②$$

From ②, $3m^2-8m-3=0$

$\therefore\ m=-\frac{1}{3},\ 3$

$\therefore\ \boldsymbol{y=-\frac{1}{3}x+5,\ y=3x-5}$



For any line/circle that passes through the point(s) of intersection of two lines/circles, we have

Let A be the point of intersection of two lines:

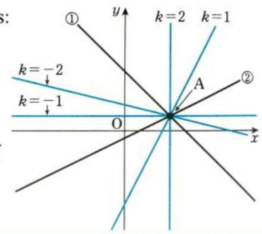
$x+y-4=0$ …①

$x-2y-1=0$ …②

Since point A lies on lines ① and ②,

$k(x+y-4)+(x-2y-1)=0$

represents a line passing through the point of intersection A regardless of the value of constant k.



Let A and B be the points of intersection of two circles:

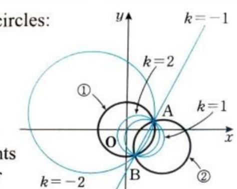
$x^2+y^2-10=0$ …①

$x^2+y^2-8x+4y+10=0$ …②

Since points A and B lie on circles ① and ②,

$k(x^2+y^2-10)+(x^2+y^2-8x+4y+10)=0$

represents a figure which passes through the points of intersection A and B regardless of the value of constant k.

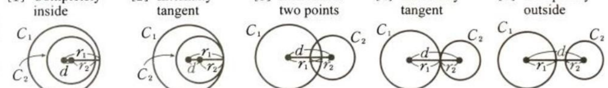


Distance between the centres of two circles

Positional Relationship between Two Circles

Let the radii of two circles C_1 and C_2 be r_1 and $r_2 (r_1>r_2)$ respectively and the distance between the centres of the circles be d.

[1] Completely inside	[2] Internally tangent	[3] Intersect at two points	[4] Externally tangent	[5] Completely outside
$d<r_1-r_2$	$d=r_1-r_2$	$r_1-r_2<d<r_1+r_2$	$d=r_1+r_2$	$d>r_1+r_2$



M 51-70 : Loci

1) Define the point (of locus) (x, y). **2)** Formulate a relationship between x and y given the information.

Given points $A(-1, -2)$ and $B(2, 4)$ where $AP : BP=1 : 2$, find the locus of point P.

Let point P be (x, y).

$AP=\sqrt{(x+1)^2+(y+2)^2}$

$BP=\sqrt{(x-2)^2+(y-4)^2}$

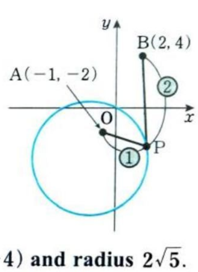
Since $AP : BP=1 : 2$, $2AP=BP$; therefore,

$2\sqrt{(x+1)^2+(y+2)^2}=\sqrt{(x-2)^2+(y-4)^2}$

$\therefore\ x^2+y^2+4x+8y=0$

$(x+2)^2+(y+4)^2=20$

Thus, the locus is **a circle with centre $(-2, -4)$ and radius $2\sqrt{5}$.**



When there are additional moving (arbitrary) points, define those points clearly and use the information that they move along a known curve.

Given moving point Q on parabola $y=x^2$, find the locus of the centre of gravity G of △ABQ with vertices $A(-1, -2)$, $B(4, -1)$ and Q.

Let point G be (x, y) and point Q be (s, t).

Since point Q lies on parabola $y=x^2$,

$t=s^2$ …①

Since point G is the centre of gravity of △ABQ,

$x=\frac{-1+4+s}{3}=\frac{s+3}{3}$, i.e. $s=3x-3$

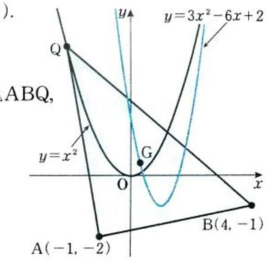
$y=\frac{-2-1+t}{3}=\frac{t-3}{3}$, i.e. $t=3y+3$

Substituting them into ①,

$3y+3=(3x-3)^2$

So, $y=3x^2-6x+2$

Therefore, the locus is **parabola $y=3x^2-6x+2$.**



Sometimes, **parametric equations** are used to represent a curve. We can form an equation involving x and y by eliminating the parameter t.

For all real numbers t, find the locus of centre P of circle $x^2+y^2-2tx+2(t-1)y=0$ …①.

From ①, $(x-t)^2+[y+(t-1)]^2=2t^2-2t+1$ …②

Also, $2t^2-2t+1=2\left(t-\frac{1}{2}\right)^2+\frac{1}{2}>0$ all

Therefore, ② represents a circle for all values of t.

Let point P be (x, y). From ②,

$x=t$ …③

$y=-t+1$ …④

$\Big\}$ parametric equations

Therefore, the locus is **line $y=-x+1$**

Sometimes, the parameter has to be eliminated at discretion (all points that make the denominator of a fraction *zero* must be excluded)

For all real numbers m, find the locus of the point of intersection $P(x, y)$ of two lines $mx+y=4m$ …① and $x-my=-4m$ …②.

From ①, $m(x-4)=-y$

(i) When $x\neq4$, $m=-\frac{y}{x-4}$

Substituting into ②, $x+\frac{y^2}{x-4}=\frac{4y}{x-4}$

So, $(x-2)^2+(y-2)^2=8$ …③

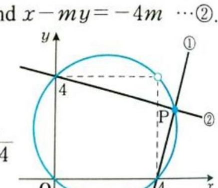
Given ③, if $x=4$, $y=0, 4$

Therefore, when $x\neq4$, points $(4, 0)$ and $(4, 4)$ are excluded in ③.

(ii) When $x=4$, from ①, $y=0$

Substituting $x=4$ and $y=0$ into ②, when $m=-1$, ② is satisfied. So, point $(4, 0)$ satisfies the condition.

From (i) and (ii), the locus is a circle with its centre at point $(2, 2)$ and radius $2\sqrt{2}$ (excluding point $(4, 4)$).



More variations: Ellipse (M68b), hyperbola (level N 2017 version)

The locus of point P, for which the sum of the distances from two fixed points F and F′ on a plane is constant, is called an ***ellipse***.

Generally, an ellipse is represented as

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>0, b>0)$